ON THE SECOND MEAN-VALUE THEOREM OF THE INTEGRAL CALCULUS

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THE second mean-value theorem in the two forms, one of which is due to Bonnet, and the other to Du Bois-Reymond and Weierstrass, is a very valuable instrument in analysis as affording a means of estimating the values of definite integrals. The theorem relates to the integral of the product of two functions $f(x)$, $\phi(x)$ defined for an interval (a, b); the first of these functions being limited and monotone in the interval. A considerable number of proofs of the theorem have been given,* of varying degrees of generality as regards the nature of the function $\phi(x)$.

In the first part of the present communication a simple proof of the theorem is given, in which the only restriction imposed upon the function $\phi(x)$ is that it possesses a Lebesgue integral in the interval (a, b).

The only other case which remains for consideration is that in which $\phi(x)$ possesses only a non-absolutely convergent improper integral in (a, b). The definition usually employed, of late years, for such integrals is that of Harnack, which is applicable to both absolutely and non-absolutely convergent integrals. It has been regarded as doubtful by various writers whether the existence of such a non-absolutely convergent integral in the interval (a, b) necessarily entails the existence of the integral of the same function in a sub-interval (a', b') contained in (a, b). For example, it was denied by Stolz that this is the case.† All doubt upon the matter was however removed by E. H. Moore,‡ who proved that, if $\int_a^b \phi(x)\,dx$ exists in accordance with Harnack's definition, then also $\int_{a'}^{b'} \phi(x)\,dx$ also exists, where

$$a \leqslant a' < b' \leqslant b.$$

* For references, see my work *Theory of Functions of a Real Variable*, pp. 359, 360.

† See *Grundzüge*, Vol. III. p. 277.

‡ *Trans. Amer. Math. Soc.*, Vol. II., p. 296 and p. 459.

He also proved that the second integral is uniformly convergent for all values of a', b'; and that the relation

$$\int_a^x \phi(x)\,dx + \int_x^b \phi(x)\,dx = \int_a^b \phi(x)\,dx$$

is valid. I have, in a former paper, introduced an extension of Harnack's definition, in which the improper integral is defined as the limit of a sequence of Lebesgue integrals, instead of that of a sequence of Riemann integrals. I have elsewhere* pointed out that E. H. Moore's results are applicable when this extension is taken instead of Harnack's original definition; and I have shewn that, in accordance with this extended definition, $\int_a^x \phi(x)$ is a continuous function of x.

In the second part of the present paper it is shewn that the existence of $\int_a^b \phi(x)\,dx$ as a non-absolutely convergent integral, in accordance with either Harnack's definition or its extension, entails as a necessary consequence the existence of $\int_a^b f(x)\,\phi(x)\,dx$, where $f(x)$ is limited and monotone in (a, b); or more generally when $f(x)$ is of limited total fluctuation (*à variation bornée*). This general result I believe to be new.† Lastly, it is shewn that the second mean-value theorem holds for the case of such a function $\phi(x)$ as possesses only a non-absolutely convergent improper integral in the interval (a, b).

1. Let $\phi(x)$ be a function which, whether it be limited or unlimited in the interval (a, b), possesses a Lebesgue integral in that interval. Let $f(x)$ be limited and monotone in (a, b), and let it never increase as x increases from a to b; and suppose it to have no negative values in the interval.

Let ϵ_r be an arbitrarily chosen positive number $< f(a+0)-f(b-0)$, and let the function $f_r(x)$ be defined for the interval (a, b) as follows :—

An interval (a, x_1) can be determined such that $f(a+0)-f(x) < \epsilon_r$, for $a \leqslant x < x_1$, and such that $f(a+0)-f(x_1) \geqslant \epsilon_r$. In case x_1 is a point of continuity of $f(x)$, we shall have $f(a+0)-f(x_1) = \epsilon_r$; but, if x_1 is point of discontinuity, we may have $f(a+0)-f(x_1) > \epsilon_r$. Next determine an interval (x_1, x_2) such that $f(x_1+0)-f(x) < \epsilon_r$, for $x_1 \leqslant x < x_2$, and that $f(x_1+0)-f(x_2) \geqslant \epsilon_r$. Proceed in this manner to determine intervals

* See "Functions of a Real Variable," p. 558.

† The special case in which the set of points of infinite discontinuity is finite is given by Dini; see *Grundlagen*, p. 424. He employs the older definition of Cauchy.

(x_2, x_3), (x_3, x_4), ... ; then for some finite value of n not exceeding $\dfrac{f(a+0)-f(b-0)}{\epsilon_r}$, the point x_n must coincide with b.

Let $f_r(x) = f(a+0)$ for $a \leqslant x < x_1$; let $f_r(x) = f(x_1+0)$ for $x_1 \leqslant x < x_2$; and, in general $f_r(x) = f(x_s+0)$ for $x_s \leqslant x < x_{s+1}$. The function $f_r(x)$ has only a finite number of values in the interval (a, b) ; it is monotone, never increases as x increases, and is never negative. Moreover, we have $0 \leqslant f_r(x)-f(x) < \epsilon_r$ for every value of x except for the values $a, x_1, x_2, \ldots, x_{n-1}, b$.

We have now

$$\int_a^b f_r(x)\,\phi(x)\,dx$$
$$= f(a+0)\int_a^{x_1}\phi(x)\,dx + f(x_1+0)\int_{x_1}^{x_2}\phi(x)\,dx + \ldots + f(x_{n-1}+0)\int_{x_{n-1}}^{b}\phi(x)\,dx.$$

Denote $\int_a^x \phi(x)\,dx$ by $F(x)$, then

$$\int_a^b f_r(x)\,\phi(x)\,dx$$
$$= f(a+0)F(x_1) + f(x_1+0)\{F(x_2)-F(x_1)\} + \ldots + f(x_{n-1}+0)\{F(b)-F(x_{n-1})\}$$
$$= \{f(a+0)-f(x_1+0)\}\,F(x_1) + \{f(x_1+0)-f(x_2+0)\}\,F(x_2) + \ldots$$
$$+ \{f(x_{n-2}+0)-f(x_{n-1}+0)\}\,F(x_{n-1}) + f(x_{n-1}+0)\,F(b).$$

Since $\quad f(a+0)-f(x_1+0),\ f(x_1+0)-f(x_2+0),\ \ldots,\ f(x_{n-1}+0)$

are all positive, the expression on the right hand will be unaltered if $F(x_1)$, $F(x_2)$, ..., $F(b)$ be all replaced by some number N which lies between the greatest and the least of these n numbers. The expression then becomes $Nf(a+0)$. Moreover, it is known that $F(x)$ is continuous in the interval (a, b), and it therefore follows that some value ξ_r of x exists such that $N = F(\xi_r)$. It has, therefore, been proved that

$$\int_a^b f_r(x)\,\phi(x)\,dx = f(a+0)\int_a^{\xi_r}\phi(x)\,dx,$$

where ξ_r is some point in the interval (a, b).

Also $$\left|\int_a^b f_r(x)\,\phi(x)\,dx - \int_a^b f(x)\,\phi(x)\,dx\right| < \epsilon_r \int_a^b |\,\phi(x)\,|\,dx\,;$$

the integral on the right-hand side being existent, because every Lebesgue integral is absolutely convergent. It follows that

$$\left|\int_a^b f(x)\,\phi(x)\,dx - f(a+0)\int_a^{\xi_r}\phi(x)\,dx\right| < \eta_r,$$

where $$\eta_r = \epsilon_r \int_a^b |\,\phi(x)\,|\,dx.$$

Let $r = 1, 2, 3, \ldots$, where $\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_r, \ldots$ is a sequence which converges to zero; also $\eta_1, \eta_2, \eta_3, \ldots, \eta_r, \ldots$ is a sequence which converges to zero. The points $\xi_1, \xi_2, \ldots, \xi_r, \ldots$ form a sequence which has at least one limiting point, and it is clear that the sequence $\{\epsilon_r\}$ may be so chosen by neglecting, if necessary, a part, that the sequence $\{\xi_r\}$ has a single limiting point $\bar{\xi}$.

We have then

$$\left| \int_a^b f(x)\,\phi(x)\,dx - f(a+0)\int_a^{\bar{\xi}} \phi(x)\,dx \right| < \eta_r + f(a+0)\left| \int_{\xi_r}^{\bar{\xi}} \phi(x)\,dx \right|.$$

If ζ be an arbitrarily chosen positive number, as small as we please, a value r_1 of r may be so chosen that $\eta_r < \frac{1}{2}\zeta$, and such that

$$f(a+0)\left| \int_{\xi_r}^{\bar{\xi}} \phi(x)\,dx \right| < \tfrac{1}{2}\zeta,$$

provided $r \geqslant r_1$. Then we have

$$\left| \int_a^b f(x)\,\phi(x)\,dx - f(a+0)\int_a^{\bar{\xi}} \phi(x)\,dx \right| < \zeta;$$

and therefore, since ζ is arbitrarily small, we must have

$$\int_a^b f(x)\,\phi(x)\,dx = f(a+0)\int_a^{\bar{\xi}} \phi(x)\,dx. \tag{1}$$

In a precisely similar manner, when $f(x)$ never diminishes as x increases from a to b, and is never negative, it may be shewn that

$$\int_a^b f(x)\,\phi(x)\,dx = f(b-0)\int_{\bar{\eta}}^b \phi(x)\,dx, \tag{2}$$

where $\bar{\eta}$ is some point in the interval (a, b).

In case

$$f(a) = f(a+0), \qquad f(b) = f(b-0),$$

these results are equivalent to Bonnet's form of the second mean-value theorem.

Next let $f(x)$ be only restricted to be limited and monotone in (a, b), but unrestricted as regards sign. In case $f(x)$ diminishes as x increases, we may apply the theorem (1) to the function $f(x) - f(b-0)$, and we thus have

$$\int_a^b f(x)\,\phi(x)\,dx = f(a+0)\int_a^{\bar{\xi}} \phi(x)\,dx + f(b-0)\int_{\bar{\xi}}^b \phi(x)\,dx.$$

In case $f(x)$ increases as x increases, we may apply the theorem (2) to the function $f(x)-f(a+0)$, and we thus have

$$\int_a^b f(x)\,\phi(x)\,dx = f(a+0)\int_a^{\bar{\eta}}\phi(x)\,dx+f(b-0)\int_{\bar{\eta}}^b\phi(x)\,dx.$$

The following theorem has now been established:—

If $f(x)$ be limited and monotone in the interval (a, b), and if $\phi(x)$ be any function, limited or unlimited, which has a Lebesgue integral in the interval (a, b), then

$$\int_a^b f(x)\,\phi(x)\,dx = f(a+0)\int_a^X\phi(x)\,dx+f(b-0)\int_X^b\phi(x)\,dx,$$

where X is some point in the interval (a, b).

In order to obtain the more general form of this theorem, let A and B be numbers such that $A \geqslant f(a+0)$, $B \leqslant f(b-0$, when $f(x)$ diminishes as x increases from a to b; or else, let $A \leqslant f(a+0)$, $B \geqslant f(b-0)$, when $f(x)$ increases as x increases from a to b.

Consider an interval $(a-\lambda, b+\lambda)$ which contains (a, b) in its interior, and let $f(x) = A$, for $a-\lambda \leqslant x < a$, and $f(x) = B$, for $b < x \leqslant b+\lambda$, the function $f(x)$ being already defined for $a \leqslant x \leqslant b$. Let $\phi(x) = 0$, for $a-\lambda \leqslant x < a$ and for $b < x \leqslant b+\lambda$, where $\phi(x)$ has already been defined for $a \leqslant x \leqslant b$. Now apply the theorem established above to the interval $(a-\lambda, b+\lambda)$, for which $f(a-\lambda+0) = A$, $f(b+\lambda-0) = B$. We then have

$$\int_a^b f(x)\,\phi(x)\,dx = A\int_a^X\phi(x)\,dx+B\int_X^b\phi(x)\,dx,$$

where X is some point in the interval $(a-\lambda, b+\lambda)$, and which clearly lies in (a, b).

This general theorem may now be stated as follows:—

If $f(x)$ be a function which is limited and monotone in the interval (a, b), and if $\phi(x)$ be any function, limited or unlimited, which has a Lebesgue integral in (a, b); then, if A, B be numbers such that

$$A \geqslant f(a+0), \qquad B \leqslant f(b-0),$$

or

$$A \leqslant f(a+0), \qquad B \geqslant f(b-0),$$

according as $f(x)$ diminishes or increases from a to b,

$$\int_a^b f(x)\,\phi(x)\,dx = A\int_a^X\phi(x)\,dx+B\int_X^b\phi(x)\,dx,$$

where X is some number in the interval (a, b). The number X will *depend on the values of A and B. In particular we may have $A = f(a)$, $B = f(b)$, or also $A = f(a+0)$, $B = f(b-0)$.*

In case the function $f(x)$ is never negative in the interval (a, b), we may take $B = 0$ if $f(x)$ is a diminishing function; and we may take $A = 0$ if $f(x)$ is an increasing function. We obtain thus the following generalization of Bonnet's theorem:—

If $f(x)$ be a limited monotone function which is never negative in the interval (a, b), and if $\phi(x)$ be any limited, or unlimited, function which has a Lebesgue integral in (a, b), then

$$\int_a^b f(x)\,\phi(x)\,dx = A\int_a^X \phi(x)\,dx,$$

where A is any number such that $A \geqslant f(a+0)$, and X is a number in the interval (a, b), dependent on A, provided $f(x)$ diminishes as x increases from a to b. Also, when $f(x)$ increases as x increases from a to b, we have

$$\int_a^b f(x)\,\phi(x)\,dx = B\int_X^b \phi(x)\,dx,$$

where B is any number $\geqslant f(b-0)$, and X is some number in the interval (a, b) dependent on the value of B. In particular, we may take $A = f(a)$, $B = f(b)$, in the two cases.

2. The mean-value theorem has been proved above for the case in which the function $\phi(x)$ is restricted only by the assumption that it possesses a Lebesgue integral in the interval (a, b). In particular, $\phi(x)$ may have a Riemann integral, or may have an absolutely convergent improper integral in accordance with the definition of Harnack. There remains for consideration only the case in which $\phi(x)$ has a non-absolutely convergent improper integral in the interval (a, b). Harnack's extension of Riemann's definition is applicable to define such improper integrals, but a wider definition is obtained by extending Harnack's definition, so that the improper integral is taken to be the limit of a sequence of Lebesgue integrals instead of that of a sequence of Riemann integrals.*

This extension of Harnack's definition, which applies both to absolutely

* I have given the extension of Harnack's definition in *The Theory of Functions of a Real Variable*, p. 557.

and to non-absolutely convergent improper integrals may be stated as follows:—

Let $\phi(x)$ be a function which has a non-dense closed set G of points of infinite discontinuity; the content of the set G being zero. Also let $\phi(x)$ be such that, in any interval whatever contained in (a, b) which contains, in its interior and at its extremities, no point of the set G, it has an integral in accordance with the definition of Lebesgue, or in particular in accordance with that of Riemann. Let the points of G be enclosed in the interiors of intervals of a finite set $\delta_1, \delta_2, \ldots, \delta_n$, so that each interval of the set contains at least one point of G. Let the remaining part of (a, b) consist of the intervals $\eta_1, \eta_2, \ldots, \eta_{\bar{n}}$ which are free in their interiors and at their ends from points of G. Let $S_{\bar{n}}$ denote the integral of $\phi(x)$ taken through the set of intervals $\{\eta\}$. Let a sequence of such sets of intervals $\{\delta\}$ be taken such that $\sum_1^n \delta$ converges to zero as n is indefinitely increased; $\bar{n}$ having the values in a sequence of numbers which increase indefinitely. If the numbers $S_{\bar{n}}$ converge, as $\bar{n}$ is indefinitely increased, to a definite number S, independent of the particular sequence of sets of intervals $\{\delta\}$ chosen, subject only to the condition

$$\lim_{n=\infty} \sum_1^n \delta = 0,$$

then the number S is defined to be the improper integral $\int_a^b \phi(x)\,dx$.

Whenever an improper integral, so defined, is absolutely convergent, the definition is in accordance with that of Lebesgue.* We need therefore consider only the case in which the integral is non-absolutely convergent. It is known† that, if $\int_a^b \phi(x)dx$ exist as a non-absolutely convergent integral, $\int_a^x \phi(x)$ also exists, and is a continuous function of x. Moreover, it is known‡ that the convergence of $\int_a^x \phi(x)dx$ is uniform for all values of x in the interval (a, b).

It will now be shewn that, if $\int_a^b \phi(x)\,dx$ exists in accordance with the above definition, or in particular in accordance with that of Harnack,

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* See *Theory of Functions of a Real Variable*, p. 397.

† *Ibid.*, p. 558.

‡ *Ibid.*, p. 383.

then $\int_a^b f(x)\,\phi(x)\,dx$ also exists; where $f(x)$ denotes as before a function which is monotone and limited in (a, b).

Let $\phi_\delta(x)$ denote a function which is equal to zero at all interior points of the intervals of a finite set $\{\delta\}$ which enclose the points of G, and which is equal to $\phi(x)$ at all points of (a, b) not in the interior of the intervals δ. Let $\phi_{\delta'}(x)$ denote the corresponding function for another such set of intervals $\{\delta'\}$. The condition of uniform convergence of $\int_a^x \phi(x)\,dx$ is expressed by the statement that, corresponding to any arbitrarily chosen positive number ϵ, a number ζ can be determined such that for any two sets of intervals $\{\delta\}$, $\{\delta'\}$ whatever, of the kind specified in the definition, and such that $\Sigma\delta < \zeta$, $\Sigma\delta' < \zeta$, the condition

$$\left| \int_a^x \phi_\delta(x)\,dx - \int_a^x \phi_{\delta'}(x)\,dx \right| < \epsilon$$

is satisfied, for all values of x in (a, b).

Let $F(x)$ denote the limited function defined by

$$F(x) \equiv \phi_\delta(x) - \phi_{\delta'}(x);$$

we may then apply the second mean-value theorem to the function $F(x)$. Thus

$$\int_a^b f(x)\,F(x)\,dx = f(a)\int_a^\xi F(x)\,dx + f(b)\int_\xi^b F(x)\,dx,$$

where ξ is some point in the interval (a, b).

We have therefore

$$\left| \int_a^b f(x)\,\phi_\delta(x)\,dx - \int_a^b f(x)\,\phi_{\delta'}(x)\,dx \right| < \epsilon\,\{|f(a)| + |f(b)|\}.$$

Denoting the expression on the right-hand side by ϵ', we see that, corresponding to the arbitrarily chosen positive number ϵ', the number ζ can be so chosen that for any two sets of intervals $\{\delta\}$, $\{\delta'\}$, such that $\Sigma\delta < \zeta$, $\Sigma\delta' < \zeta$, the condition

$$\left| \int_a^b f(x)\,\phi_\delta(x)\,dx - \int_a^b f(x)\,\phi_{\delta'}(x)\,dx \right| < \epsilon'$$

is satisfied. This is, however, the necessary and sufficient condition for the existence of $\int_a^b f(x)\,\phi(x)\,dx$, in accordance with the above definition.

The following theorem has now been established :—

If $\phi(x)$ have an improper integral in (a, b), either absolutely or non-absolutely convergent, in accordance with the above definition, or in *particular, in accordance with the definition of Harnack, and if $f(x)$ be any limited and monotone function defined for the same interval, then $f(x)\,\phi(x)$ also has an improper integral in (a, b) in accordance with the same definition.*

Since any function of limited total fluctuation is expressible as the difference of two monotone functions $f_1(x)$, $f_2(x)$, and since the two functions $f_1(x)\,\phi(x)$, $f_2(x)\,\phi(x)$ have the same set G of points of infinite discontinuity as $\phi(x)$ has, we obtain the following general theorem :—

If $\phi(x)$ have an improper integral in (a, b), either absolutely or non-absolutely convergent, and if $f(x)$ be any function with limited total fluctuation (à variation bornée) in (a, b), then $\int_a^b f(x)\,\phi(x)\,dx$ exists as an improper integral.

This theorem is, of course, well known for the case in which $\int_a^b \phi(x)\,dx$ is absolutely convergent, but is, in its generality, so far as I know, new for the case in which the integral of $\phi(x)$ is non-absolutely convergent.

It will be found useful in deciding as to the existence of non-absolutely convergent integrals of special functions. For example, if the Fourier coefficient $\frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(x)\,dx$, corresponding to $\phi(x)$, exists as a non-absolutely convergent improper integral, then all the other coefficients $\frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\cos nx\,dx$, $\frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\sin nx\,dx$ necessarily exist.

3. It will now be shewn that the second mean-value theorem holds for any function $\phi(x)$ which has a non-absolutely convergent improper integral in (a, b).

Applying the mean-value theorem to the limited function $\phi_\delta(x)$, we have

$$\int_a^b f(x)\,\phi_\delta(x)\,dx = A\int_a^{X_\delta} \phi_\delta(x)\,dx + B\int_{X_\delta}^b \phi_\delta(x)\,dx,$$

where A and B are subject to the same conditions as in § 1. Now

$$\int_a^{X_\delta} \phi_\delta(x)\,dx - \int_a^{X_\delta} \phi(x)\,dx, \qquad \int_{X_\delta}^b \phi_\delta(x)\,dx - \int_{X_\delta}^b \phi(x)\,dx,$$

are both numerically less than an arbitrarily chosen number ϵ, provided $\Sigma\delta$ is sufficiently small. This follows from the uniform convergence of $\int_a^x \phi(x)\,dx$.

Also $\int_a^b f(x)\,\phi_\delta(x)\,dx$ differs from $\int_a^b f(x)\,\phi(x)\,dx$ by less than ϵ, if $\Sigma\delta$ is sufficiently small. Hence we have

$$\int_a^b f(x)\,\phi(x)\,dx = A\int_a^{X_\delta}\phi(x)\,dx + B\int_{X_\delta}^b \phi(x)\,dx + \eta,$$

where $|\eta|$ is arbitrarily small. By similar reasoning to that employed at the end of § 1, it follows from the continuity of $\int_a^{X_\delta}\phi(x)\,dx$, $\int_{X_\delta}^b \phi(x)\,dx$ with respect to x, that a number ξ in (a, b) exists, such that

$$\int_a^b f(x)\,\phi(x)\,dx = A\int_a^{\xi}\phi(x)\,dx + B\int_{\xi}^b \phi(x)\,dx.$$

Bonnet's form of the theorem may be deduced as in § 1. The complete generality of the second mean-value theorem has accordingly been established.