> # This worksheet underpins the 'Schur' calculations in James Davenport's paper 'Without Loss of Generality'

> $Schur := a^k \cdot (a-b) \cdot (a-c) + b^k \cdot (b-a) \cdot (b-c) + c^k \cdot (c-a) \cdot (c-b);$

$$Schur := a^k \, (a-b) \, (a-c) + b^k \, (b-a) \, (b-c) + c^k \, (c-a) \, (c-b) \tag{1}$$

> $Schur2 := eval(Schur, k=2);$

$$Schur2 := a^2 \, (a-b) \, (a-c) + b^2 \, (b-a) \, (b-c) + c^2 \, (c-a) \, (c-b) \tag{2}$$

> $with(RegularChains);$

$[AlgebraicGeometryTools, ChainTools, ConstructibleSetTools, Display,$ **(3)**
    $DisplayPolynomialRing, Equations, ExtendedRegularGcd, FastArithmeticTools,$
    $Inequations, Info, Initial, Intersect, Inverse, IsRegular, LazyRealTriangularize, MainDegree,$
    $MainVariable, MatrixCombine, MatrixTools, NormalForm, ParametricSystemTools,$
    $PolynomialRing, Rank, RealTriangularize, RegularGcd, RegularizeInitial, SamplePoints,$
    $SemiAlgebraicSetTools, Separant, SparsePseudoRemainder, SuggestVariableOrder, Tail,$
    $Triangularize]$

> $with(SemiAlgebraicSetTools);$

$[BoxValues, Complement, CylindricalAlgebraicDecompose, Difference, DisplayParametricBox,$ **(4)**
    $DisplayQuantifierFreeFormula, EmptySemiAlgebraicSet, Intersection, IsContained,$
    $IsEmpty, IsParametricBox, LinearSolve, PartialCylindricalAlgebraicDecomposition,$
    $PositiveInequalities, Projection, QuantifierElimination, RealRootCounting,$
    $RealRootIsolate, RefineBox, RefineListBox, RemoveRedundantComponents,$
    $RepresentingBox, RepresentingChain, RepresentingQuantifierFreeFormula,$
    $RepresentingRootIndex, SignAtBox, VariableOrdering]$

> $R := PolynomialRing([a, b, c]);$

$$R := polynomial\_ring \tag{5}$$

> **read**("C:\\Users\\James\\Shared\\Triangular\\trunk\\Maple\\ProjectionCAD.mpl")
    # *Obtainable from http://opus.bath.ac.uk/43911/*

"This is V3.18 of the ProjectionCAD module from 11th February 2015, designed and tested for   **(6)**
    use in Maple 18."

> $with(ProjectionCAD);$

$[CADDist, CADFull, CADGenerateStack, CADLifting, CADNormDist, CADProjection,$ **(7)**
    $ECCAD, ECCADFormulations, ECCADHeuristic, ECCADProjFactors, ECCADProjOp,$
    $LCAD, LCADDisplay, LCADRecursive, LTTICAD, LVCAD, LVTTICAD, NumCellsInCAD,$
    $NumCellsInPiecewiseCAD, TTICAD, TTICADDist, TTICADFormulations,$
    $TTICADHeuristic, TTICADNormDist, TTICADProjFactors, TTICADProjOp,$
    $TTICADQFFFormulations, TTICADQFFHeuristic, TTICADResCAD, TTICADResCADSet,$
    $VCAD, VCADLiftOverLowCAD, VTTICAD, VariableOrderingHeuristic,$
    $VariableOrderings, ndrr, sotd]$

> $Z := CADFull([expand(Schur2)], R[variables], output=piecewise); nops(\%);$
    # *computed via ProjectionLifting*

Warning, no method was specified, McCallum's algorithm will be
used

$$Z := \begin{cases}
\begin{cases}
\begin{cases}
[regular\_chain, [[-1,-1],[-2,-2],[0,0]]] & b < c \\
\begin{cases}
[regular\_chain, [[-1,-1],[-1,-1],[-2,-2]]] & a < c \\
[regular\_chain, [[-1,-1],[-1,-1],[-1,-1]]] & a = c \\
\left[regular\_chain, \left[[-1,-1],[-1,-1],\left[-\frac{1}{2},-\frac{1}{2}\right]\right]\right] & c < a < 0 \\
[regular\_chain, [[-1,-1],[-1,-1],[0,0]]] & a = 0 \\
[regular\_chain, [[-1,-1],[-1,-1],[1,1]]] & 0 < a
\end{cases} & b = c \\
\left[regular\_chain, \left[[-1,-1],\left[-\frac{1}{2},-\frac{1}{2}\right],[0,0]\right]\right] & c < b < 0 \\
\begin{cases}
[regular\_chain, [[-1,-1],[0,0],[-2,-2]]] & a < c \\
[regular\_chain, [[-1,-1],[0,0],[-1,-1]]] & a = c \\
[regular\_chain, [[-1,-1],[0,0],[0,0]]] & c < a
\end{cases} & b = 0 \\
[regular\_chain, [[-1,-1],[1,1],[0,0]]] & 0 < b
\end{cases} & c < 0 \\[4pt]
\begin{cases}
\begin{cases}
[regular\_chain, [[0,0],[-1,-1],[-2,-2]]] & a < b \\
[regular\_chain, [[0,0],[-1,-1],[-1,-1]]] & a = b \\
[regular\_chain, [[0,0],[-1,-1],[0,0]]] & b < a
\end{cases} & b < 0 \\
\begin{cases}
[regular\_chain, [[0,0],[0,0],[-1,-1]]] & a < 0 \\
[regular\_chain, [[0,0],[0,0],[0,0]]] & a = 0 \\
[regular\_chain, [[0,0],[0,0],[1,1]]] & 0 < a
\end{cases} & b = 0 \\
\begin{cases}
[regular\_chain, [[0,0],[1,1],[0,0]]] & a < b \\
[regular\_chain, [[0,0],[1,1],[1,1]]] & a = b \\
[regular\_chain, [[0,0],[1,1],[2,2]]] & b < a
\end{cases} & 0 < b
\end{cases} & c = 0 \\[4pt]
\begin{cases}
[regular\_chain, [[1,1],[-1,-1],[0,0]]] & b < 0 \\
\begin{cases}
[regular\_chain, [[1,1],[0,0],[0,0]]] & a < c \\
[regular\_chain, [[1,1],[0,0],[1,1]]] & a = c \\
[regular\_chain, [[1,1],[0,0],[2,2]]] & c < a
\end{cases} & b = 0 \\
\left[regular\_chain, \left[[1,1],\left[\frac{1}{2},\frac{1}{2}\right],[0,0]\right]\right] & 0 < b < c \\
\begin{cases}
[regular\_chain, [[1,1],[1,1],[-1,-1]]] & a < 0 \\
[regular\_chain, [[1,1],[1,1],[0,0]]] & a = 0 \\
\left[regular\_chain, \left[[1,1],[1,1],\left[\frac{1}{2},\frac{1}{2}\right]\right]\right] & 0 < a < c
\end{cases} & b = c
\end{cases} & 0 < c
\end{cases}$$

(8)

> $CADSchur2 := CylindricalAlgebraicDecompose([\,expand(Schur2)\,], R, output = piecewise,$
$method = recursive); nops(\%); \quad \# \ computed \ via \ regular \ chains$

$$
CADSchur2 := \begin{cases}
\begin{cases}
1 & b < c \\[4pt]
\begin{cases}
1 & a < c \\
1 & a = c \\
1 & c < a < 0 \\
1 & a = 0 \\
1 & 0 < a
\end{cases} & b = c \\[20pt]
1 & c < b < 0 \\[4pt]
\begin{cases}
1 & a < RootOf\!\left(\_Z^3 - c^3,\, index = real_1\right) \\
1 & a = RootOf\!\left(\_Z^3 - c^3,\, index = real_1\right) \\
1 & RootOf\!\left(\_Z^3 - c^3,\, index = real_1\right) < a
\end{cases} & b = 0 \\[20pt]
1 & 0 < b
\end{cases} & c < 0 \\[60pt]
\begin{cases}
\begin{cases}
1 & a < RootOf\!\left(\_Z^3 - b^3,\, index = real_1\right) \\
1 & a = RootOf\!\left(\_Z^3 - b^3,\, index = real_1\right) \\
1 & RootOf\!\left(\_Z^3 - b^3,\, index = real_1\right) < a
\end{cases} & b < 0 \\[20pt]
\begin{cases}
1 & a < 0 \\
1 & a = 0 \\
1 & 0 < a
\end{cases} & b = 0 \\[16pt]
\begin{cases}
1 & a < RootOf\!\left(\_Z^3 - b^3,\, index = real_1\right) \\
1 & a = RootOf\!\left(\_Z^3 - b^3,\, index = real_1\right) \\
1 & RootOf\!\left(\_Z^3 - b^3,\, index = real_1\right) < a
\end{cases} & 0 < b
\end{cases} & c = 0 \\[60pt]
\begin{cases}
1 & b < 0 \\[4pt]
\begin{cases}
1 & a < RootOf\!\left(\_Z^3 - c^3,\, index = real_1\right) \\
1 & a = RootOf\!\left(\_Z^3 - c^3,\, index = real_1\right) \\
1 & RootOf\!\left(\_Z^3 - c^3,\, index = real_1\right) < a
\end{cases} & b = 0 \\[20pt]
1 & 0 < b < c \\[4pt]
\begin{cases}
1 & a < 0
\end{cases} &
\end{cases} & 0 < c
\end{cases}
$$

(9)

> $Display(CADSchur2, R)$;

(10)

$$
\left[\left[\left\{ \begin{array}{ll} 1 & b < c \\ \left\{ \begin{array}{ll} 1 & a < c \\ 1 & a = c \\ 1 & c < a < 0 \\ 1 & a = 0 \\ 1 & 0 < a \end{array} \right. & b = c \\ 1 & c < b < 0 \end{array} \right. \quad c < 0 \right. \right.
$$

$$
\left[\begin{array}{ll} 1 & a < \mathit{RootOf}\left(\_Z^3 - c^3,\, \mathit{index} = \mathit{real}_1\right) \\ 1 & a = \mathit{RootOf}\left(\_Z^3 - c^3,\, \mathit{index} = \mathit{real}_1\right) \\ 1 & \mathit{RootOf}\left(\_Z^3 - c^3,\, \mathit{index} = \mathit{real}_1\right) < a \end{array} \right. \quad b = 0
$$

$$
1 \qquad 0 < b
$$

$$
\left[\left[\begin{array}{ll} 1 & a < \mathit{RootOf}\left(\_Z^3 - b^3,\, \mathit{index} = \mathit{real}_1\right) \\ 1 & a = \mathit{RootOf}\left(\_Z^3 - b^3,\, \mathit{index} = \mathit{real}_1\right) \\ 1 & \mathit{RootOf}\left(\_Z^3 - b^3,\, \mathit{index} = \mathit{real}_1\right) < a \end{array} \right. \quad b < 0 \right.
$$

$$
\left\{ \begin{array}{ll} 1 & a < 0 \\ 1 & a = 0 \\ 1 & 0 < a \end{array} \right. \quad b = 0 \qquad c = 0 \tag{10}
$$

$$
\left[\begin{array}{ll} 1 & a < \mathit{RootOf}\left(\_Z^3 - b^3,\, \mathit{index} = \mathit{real}_1\right) \\ 1 & a = \mathit{RootOf}\left(\_Z^3 - b^3,\, \mathit{index} = \mathit{real}_1\right) \\ 1 & \mathit{RootOf}\left(\_Z^3 - b^3,\, \mathit{index} = \mathit{real}_1\right) < a \end{array} \right. \quad 0 < b
$$

$$
\left[\begin{array}{ll} 1 & b < 0 \\ \left[\begin{array}{ll} 1 & a < \mathit{RootOf}\left(\_Z^3 - c^3,\, \mathit{index} = \mathit{real}_1\right) \\ 1 & a = \mathit{RootOf}\left(\_Z^3 - c^3,\, \mathit{index} = \mathit{real}_1\right) \\ 1 & \mathit{RootOf}\left(\_Z^3 - c^3,\, \mathit{index} = \mathit{real}_1\right) < a \end{array} \right. & b = 0 \\ 1 & 0 < b < c \end{array} \right. \quad 0 < c
$$

$$
\left[\begin{array}{ll} 1 & a < 0 \end{array} \right.
$$

> *Schur3* := *eval*(*Schur*, *k* = 3);

$$Schur3 := a^3\,(a-b)\,(a-c) + b^3\,(b-a)\,(b-c) + c^3\,(c-a)\,(c-b) \qquad \textbf{(11)}$$

> *Z3* := *CADFull*([*expand*(*Schur3*)], *R*[*variables*], *output* = *piecewise*); *nops*(%);

Warning, no method was specified, McCallum's algorithm will be used

$[\mathit{regular\_chain}, [[-1, -1], [-2, -$

$\left[\mathit{regular\_chain}, \left[[-1, -1], [-2, -2], \left[\frac{75}{52}\right.\right.\right.$

$[\mathit{regular\_chain}, [[-1, -1], [-2, -$

$\left[\mathit{regular\_chain}, \left[[-1, -1], \left[-\frac{1}{2}, --\right.\right.\right.$

$\left[\mathit{regular\_chain}, \left[[-1, -1], \left[-\frac{1}{2}, -\frac{1}{2}\right], \left[-\right.\right.\right.$

$\left[\mathit{regular\_chain}, \left[[-1, -1], \left[-\frac{1}{2}, -\right.\right.\right.$

$[\mathit{regular\_chain}, [[-1, -1], [1, 1], [-3, -3]]]$

$\left[\mathit{regular\_chain}, \left[[-1, -1], [1, 1], \left[-\frac{3356461}{2097152}, -\frac{839115}{524288}\right]\right]\right]$

$\left[\mathit{regular\_chain}, \left[[-1, -1], [1, 1], \left[-\frac{839115}{1048576}, -\frac{839115}{1048576}\right]\right]\right]$ $\quad RootOf\left(\_Z^5 + (-b-c)\_Z^4 +\right.$

$[\mathit{regular\_chain}, [[-1, -1], [1, 1], [0, 0]]]$

$\left[\mathit{regular\_chain}, \left[[-1, -1], [1, 1], \left[\frac{839115}{1048576}, \frac{839115}{1048576}\right]\right]\right]$ $\quad RootOf\left(\_Z^5 + (-b-c)\_Z^4 +\right.$

$\left[\mathit{regular\_chain}, \left[[-1, -1], [1, 1], \left[\frac{839115}{524288}, \frac{1678231}{1048576}\right]\right]\right]$

$[\mathit{regular\_chain}, [[-1, -1], [1, 1], [3, 3]]]$

> $Zlist := CADFull([expand(Schur2)], R[variables], output = list); nops(\%);$
Warning, no method was specified, McCallum's algorithm will be used

$Zlist := \big[[[1, 1, 1], [regular\_chain, [[-1, -1], [-2, -2], [0, 0]]]], [[1, 2, 1],$

$[regular\_chain, [[-1, -1], [-1, -1], [-2, -2]]]], [[1, 2, 2], [regular\_chain, [[-1, -1],$

$[-1, -1], [-1, -1]]]], \big[[1, 2, 3], \big[regular\_chain, \big[[-1, -1], [-1, -1], \big[-\frac{1}{2}, -\frac{1}{2}\big]\big]\big]\big],$

$[[1, 2, 4], [regular\_chain, [[-1, -1], [-1, -1], [0, 0]]]], [[1, 2, 5], [regular\_chain, [[$

$-1, -1], [-1, -1], [1, 1]]]], \big[[1, 3, 1], \big[regular\_chain, \big[[-1, -1], \big[-\frac{1}{2}, -\frac{1}{2}\big], [0,$

$0]\big]\big]\big], [[1, 4, 1], [regular\_chain, [[-1, -1], [0, 0], [-2, -2]]]], [[1, 4, 2],$

$[regular\_chain, [[-1, -1], [0, 0], [-1, -1]]]], [[1, 4, 3], [regular\_chain, [[-1, -1],$

$[0, 0], [0, 0]]]], [[1, 5, 1], [regular\_chain, [[-1, -1], [1, 1], [0, 0]]]], [[2, 1, 1],$

$[regular\_chain, [[0, 0], [-1, -1], [-2, -2]]]], [[2, 1, 2], [regular\_chain, [[0, 0], [-1,$

$-1], [-1, -1]]]], [[2, 1, 3], [regular\_chain, [[0, 0], [-1, -1], [0, 0]]]], [[2, 2, 1],$

$[regular\_chain, [[0, 0], [0, 0], [-1, -1]]]], [[2, 2, 2], [regular\_chain, [[0, 0], [0, 0],$

$[0, 0]]]], [[2, 2, 3], [regular\_chain, [[0, 0], [0, 0], [1, 1]]]], [[2, 3, 1], [regular\_chain,$

$[[0, 0], [1, 1], [0, 0]]]], [[2, 3, 2], [regular\_chain, [[0, 0], [1, 1], [1, 1]]]], [[2, 3, 3],$

$[regular\_chain, [[0, 0], [1, 1], [2, 2]]]], [[3, 1, 1], [regular\_chain, [[1, 1], [-1, -1],$

$[0, 0]]]], [[3, 2, 1], [regular\_chain, [[1, 1], [0, 0], [0, 0]]]], [[3, 2, 2], [regular\_chain,$

$[[1, 1], [0, 0], [1, 1]]]], [[3, 2, 3], [regular\_chain, [[1, 1], [0, 0], [2, 2]]]], \big[[3, 3, 1],$

$\big[regular\_chain, \big[[1, 1], \big[\frac{1}{2}, \frac{1}{2}\big], [0, 0]\big]\big]\big], [[3, 4, 1], [regular\_chain, [[1, 1], [1, 1], [$

$-1, -1]]]], [[3, 4, 2], [regular\_chain, [[1, 1], [1, 1], [0, 0]]]], \big[[3, 4, 3],$

$\big[regular\_chain, \big[[1, 1], [1, 1], \big[\frac{1}{2}, \frac{1}{2}\big]\big]\big]\big], [[3, 4, 4], [regular\_chain, [[1, 1], [1, 1],$

$[1, 1]]]], [[3, 4, 5], [regular\_chain, [[1, 1], [1, 1], [2, 2]]]], [[3, 5, 1], [regular\_chain,$

$[[1, 1], [2, 2], [0, 0]]]]\big]$

(13)

> $Z3list := CADFull([expand(Schur3)], R[variables], output = list); nops(\%);$
<u>Warning, no method was specified, McCallum's algorithm will be</u>
<u>used</u>

$Z3list := \Big[[[1, 1, 1], [regular\_chain, [[-1, -1], [-2, -2], [0, 0]]]], \Big[[1, 1, 2],$

$\Big[regular\_chain, \Big[[-1, -1], [-2, -2], \Big[\frac{755115}{524288}, \frac{1510231}{1048576} \Big] \Big] \Big] \Big], [[1, 1, 3],$

$[regular\_chain, [[-1, -1], [-2, -2], [3, 3]]]], [[1, 2, 1], [regular\_chain, [[-1, -1], [$

$-1, -1], [-2, -2]]]], [[1, 2, 2], [regular\_chain, [[-1, -1], [-1, -1], [-1, -1]]]], \Big[[1,$

$2, 3], \Big[regular\_chain, \Big[[-1, -1], [-1, -1], \Big[-\frac{1}{2}, -\frac{1}{2} \Big] \Big] \Big] \Big], [[1, 2, 4], [regular\_chain, [[$

$-1, -1], [-1, -1], [0, 0]]]], [[1, 2, 5], [regular\_chain, [[-1, -1], [-1, -1], [1, 1]]]],$

$\Big[[1, 3, 1], \Big[regular\_chain, \Big[[-1, -1], \Big[-\frac{1}{2}, -\frac{1}{2} \Big], [-1, -1] \Big] \Big] \Big], \Big[[1, 3, 2],$

$\Big[regular\_chain, \Big[[-1, -1], \Big[-\frac{1}{2}, -\frac{1}{2} \Big], \Big[\frac{755115}{1048576}, \frac{188779}{262144} \Big] \Big] \Big] \Big], \Big[[1, 3, 3],$

$\Big[regular\_chain, \Big[[-1, -1], \Big[-\frac{1}{2}, -\frac{1}{2} \Big], [2, 2] \Big] \Big] \Big], [[1, 4, 1], [regular\_chain, [[-1, -1],$

$[0, 0], [-2, -2]]]], [[1, 4, 2], [regular\_chain, [[-1, -1], [0, 0], [-1, -1]]]], [[1, 4, 3],$

$[regular\_chain, [[-1, -1], [0, 0], [0, 0]]]], [[1, 4, 4], [regular\_chain, [[-1, -1], [0,$

$0], [1, 1]]]], [[1, 4, 5], [regular\_chain, [[-1, -1], [0, 0], [2, 2]]]], [[1, 5, 1],$

$[regular\_chain, [[-1, -1], [1, 1], [-3, -3]]]], \Big[[1, 5, 2], \Big[regular\_chain, \Big[[-1, -1],$

$[1, 1], \Big[-\frac{3356461}{2097152}, -\frac{839115}{524288} \Big] \Big] \Big] \Big], \Big[[1, 5, 3], \Big[regular\_chain, \Big[[-1, -1], [1, 1], \Big[$

$-\frac{839115}{1048576}, -\frac{839115}{1048576} \Big] \Big] \Big] \Big], [[1, 5, 4], [regular\_chain, [[-1, -1], [1, 1], [0, 0]]]],$

$\Big[[1, 5, 5], \Big[regular\_chain, \Big[[-1, -1], [1, 1], \Big[\frac{839115}{1048576}, \frac{839115}{1048576} \Big] \Big] \Big] \Big], \Big[[1, 5, 6],$

$\Big[regular\_chain, \Big[[-1, -1], [1, 1], \Big[\frac{839115}{524288}, \frac{1678231}{1048576} \Big] \Big] \Big] \Big], [[1, 5, 7], [regular\_chain,$

$[[-1, -1], [1, 1], [3, 3]]]], [[2, 1, 1], [regular\_chain, [[0, 0], [-1, -1], [-2, -2]]]],$

$[[2, 1, 2], [regular\_chain, [[0, 0], [-1, -1], [-1, -1]]]], [[2, 1, 3], [regular\_chain,$

$[[0, 0], [-1, -1], [0, 0]]]], [[2, 1, 4], [regular\_chain, [[0, 0], [-1, -1], [1, 1]]]], [[2,$

$1, 5], [regular\_chain, [[0, 0], [-1, -1], [2, 2]]]], [[2, 2, 1], [regular\_chain, [[0, 0], [0, 0], [-1, -1]]]], [[2, 2, 2], [regular\_chain, [[0, 0], [0, 0], [0, 0]]]], [[2, 2, 3], [regular\_chain, [[0, 0], [0, 0], [1, 1]]]], [[2, 3, 1], [regular\_chain, [[0, 0], [1, 1], [-2, -2]]]], [[2, 3, 2], [regular\_chain, [[0, 0], [1, 1], [-1, -1]]]], [[2, 3, 3], [regular\_chain, [[0, 0], [1, 1], [0, 0]]]], [[2, 3, 4], [regular\_chain, [[0, 0], [1, 1], [1, 1]]]], [[2, 3, 5], [regular\_chain, [[0, 0], [1, 1], [2, 2]]]], [[3, 1, 1], [regular\_chain, [[1, 1], [-1, -1], [-3, -3]]]], \left[[3, 1, 2], \left[regular\_chain, \left[[1, 1], [-1, -1], \left[-\frac{3356461}{2097152}, -\frac{839115}{524288} \right] \right] \right] \right], \left[[3, 1, 3], \left[regular\_chain, \left[[1, 1], [-1, -1], \left[-\frac{839115}{1048576}, -\frac{839115}{1048576} \right] \right] \right] \right], [[3, 1, 4], [regular\_chain, [[1, 1], [-1, -1], [0, 0]]]], \left[[3, 1, 5], \left[regular\_chain, \left[[1, 1], [-1, -1], \left[\frac{839115}{1048576}, \frac{839115}{1048576} \right] \right] \right] \right], \left[[3, 1, 6], \left[regular\_chain, \left[[1, 1], [-1, -1], \left[\frac{839115}{524288}, \frac{1678231}{1048576} \right] \right] \right] \right], [[3, 1, 7], [regular\_chain, [[1, 1], [-1, -1], [3, 3]]]], [[3, 2, 1], [regular\_chain, [[1, 1], [0, 0], [-2, -2]]]], [[3, 2, 2], [regular\_chain, [[1, 1], [0, 0], [-1, -1]]]], [[3, 2, 3], [regular\_chain, [[1, 1], [0, 0], [0, 0]]]], [[3, 2, 4], [regular\_chain, [[1, 1], [0, 0], [1, 1]]]], [[3, 2, 5], [regular\_chain, [[1, 1], [0, 0], [2, 2]]]], \left[[3, 3, 1], \left[regular\_chain, \left[[1, 1], \left[\frac{1}{2}, \frac{1}{2} \right], [-2, -2] \right] \right] \right], \left[[3, 3, 2], \left[regular\_chain, \left[[1, 1], \left[\frac{1}{2}, \frac{1}{2} \right], \left[-\frac{188779}{262144}, -\frac{755115}{1048576} \right] \right] \right] \right], \left[[3, 3, 3], \left[regular\_chain, \left[[1, 1], \left[\frac{1}{2}, \frac{1}{2} \right], [1, 1] \right] \right] \right], [[3, 4, 1], [regular\_chain, [[1, 1], [1, 1], [-1, -1]]]], [[3, 4, 2], [regular\_chain, [[1, 1], [1, 1], [0, 0]]]], \left[[3, 4, 3], \left[regular\_chain, \left[[1, 1], [1, 1], \left[\frac{1}{2}, \frac{1}{2} \right] \right] \right] \right], [[3, 4, 4], [regular\_chain, [[1, 1], [1, 1], [1, 1]]]], [[3, 4, 5], [regular\_chain, [[1, 1], [1, 1], [2, 2]]]], [[3, 5, 1], [regular\_chain, [[1, 1], [2, 2], [-3, -3]]]], \left[[3, 5, 2], \left[regular\_chain, \left[[1, 1], [2, 2], \left[-\frac{1510231}{1048576}, -\frac{755115}{524288} \right] \right] \right] \right], [[3, 5, 3], [regular\_chain, [[1, 1], [2, 2], [0, 0]]]] \right]$

> $Schur4 := eval(Schur, k = 4); Z4list := CADFull([expand(Schur4)], R[variables], output = list); nops(\%);$

$$Schur4 := a^4 (a - b) (a - c) + b^4 (b - a) (b - c) + c^4 (c - a) (c - b)$$

Warning, no method was specified, McCallum's algorithm will be used

$Z4list := \Big[[[1, 1, 1], [regular\_chain, [[-1, -1], [-2, -2], [0, 0]]]], [[1, 2, 1],$

$[regular\_chain, [[-1, -1], [-1, -1], [-2, -2]]]], [[1, 2, 2], [regular\_chain, [[-1, -1],$

$[-1, -1], [-1, -1]]]], \Big[[1, 2, 3], \Big[regular\_chain, \Big[[-1, -1], [-1, -1], \Big[-\frac{1}{2}, -\frac{1}{2} \Big] \Big] \Big] \Big],$

$[[1, 2, 4], [regular\_chain, [[-1, -1], [-1, -1], [0, 0]]]], [[1, 2, 5], [regular\_chain, [[$

$-1, -1], [-1, -1], [1, 1]]]], \Big[[1, 3, 1], \Big[regular\_chain, \Big[[-1, -1], \Big[-\frac{1}{2}, -\frac{1}{2} \Big], [0,$

$0] \Big] \Big] \Big], [[1, 4, 1], [regular\_chain, [[-1, -1], [0, 0], [-2, -2]]]], [[1, 4, 2],$

$[regular\_chain, [[-1, -1], [0, 0], [-1, -1]]]], [[1, 4, 3], [regular\_chain, [[-1, -1],$

$[0, 0], [0, 0]]]], [[1, 5, 1], [regular\_chain, [[-1, -1], [1, 1], [0, 0]]]], [[2, 1, 1],$

$[regular\_chain, [[0, 0], [-1, -1], [-2, -2]]]], [[2, 1, 2], [regular\_chain, [[0, 0], [-1,$

$-1], [-1, -1]]]], [[2, 1, 3], [regular\_chain, [[0, 0], [-1, -1], [0, 0]]]], [[2, 2, 1],$

$[regular\_chain, [[0, 0], [0, 0], [-1, -1]]]], [[2, 2, 2], [regular\_chain, [[0, 0], [0, 0],$

$[0, 0]]]], [[2, 2, 3], [regular\_chain, [[0, 0], [0, 0], [1, 1]]]], [[2, 3, 1], [regular\_chain,$

$[[0, 0], [1, 1], [0, 0]]]], [[2, 3, 2], [regular\_chain, [[0, 0], [1, 1], [1, 1]]]], [[2, 3, 3],$

$[regular\_chain, [[0, 0], [1, 1], [2, 2]]]], [[3, 1, 1], [regular\_chain, [[1, 1], [-1, -1],$

$[0, 0]]]], [[3, 2, 1], [regular\_chain, [[1, 1], [0, 0], [0, 0]]]], [[3, 2, 2], [regular\_chain,$

$[[1, 1], [0, 0], [1, 1]]]], [[3, 2, 3], [regular\_chain, [[1, 1], [0, 0], [2, 2]]]], \Big[[3, 3, 1],$

$\Big[regular\_chain, \Big[[1, 1], \Big[\frac{1}{2}, \frac{1}{2} \Big], [0, 0] \Big] \Big] \Big], [[3, 4, 1], [regular\_chain, [[1, 1], [1, 1], [$

$-1, -1]]]], [[3, 4, 2], [regular\_chain, [[1, 1], [1, 1], [0, 0]]]], \Big[[3, 4, 3],$

$\Big[regular\_chain, \Big[[1, 1], [1, 1], \Big[\frac{1}{2}, \frac{1}{2} \Big] \Big] \Big] \Big], [[3, 4, 4], [regular\_chain, [[1, 1], [1, 1],$

$[1, 1]]]], [[3, 4, 5], [regular\_chain, [[1, 1], [1, 1], [2, 2]]]], [[3, 5, 1], [regular\_chain,$
$[[1, 1], [2, 2], [0, 0]]]]]]$

<div align="right">31   (15)</div>

> $Schur5 := eval(Schur, k = 5); Z4list := CADFull([expand(Schur5)], R[variables], output$
> $= list); nops(\%);$

$$Schur5 := a^5 (a - b) (a - c) + b^5 (b - a) (b - c) + c^5 (c - a) (c - b)$$

```
Warning, no method was specified, McCallum's algorithm will be
used
```

$Z4list := \Big[[[1, 1, 1], [regular\_chain, [[-1, -1], [-2, -2], [0, 0]]]], \Big[[1, 1, 2],$

$\quad \Big[regular\_chain, \Big[[-1, -1], [-2, -2], \Big[\dfrac{1719221}{1048576}, \dfrac{859611}{524288}\Big]\Big]\Big]\Big], [[1, 1, 3],$

$\quad [regular\_chain, [[-1, -1], [-2, -2], [3, 3]]]], [[1, 2, 1], [regular\_chain, [[-1, -1], [$

$\quad -1, -1], [-2, -2]]]], [[1, 2, 2], [regular\_chain, [[-1, -1], [-1, -1], [-1, -1]]]], \Big[[1,$

$\quad 2, 3], \Big[regular\_chain, \Big[[-1, -1], [-1, -1], \Big[-\dfrac{1}{2}, -\dfrac{1}{2}\Big]\Big]\Big]\Big], [[1, 2, 4], [regular\_chain, [[$

$\quad -1, -1], [-1, -1], [0, 0]]]], [[1, 2, 5], [regular\_chain, [[-1, -1], [-1, -1], [1, 1]]]],$

$\quad \Big[[1, 3, 1], \Big[regular\_chain, \Big[[-1, -1], \Big[-\dfrac{1}{2}, -\dfrac{1}{2}\Big], [-1, -1]\Big]\Big]\Big], \Big[[1, 3, 2],$

$\quad \Big[regular\_chain, \Big[[-1, -1], \Big[-\dfrac{1}{2}, -\dfrac{1}{2}\Big], \Big[\dfrac{429805}{524288}, \dfrac{859611}{1048576}\Big]\Big]\Big]\Big], \Big[[1, 3, 3],$

$\quad \Big[regular\_chain, \Big[[-1, -1], \Big[-\dfrac{1}{2}, -\dfrac{1}{2}\Big], [2, 2]\Big]\Big]\Big], [[1, 4, 1], [regular\_chain, [[-1, -1],$

$\quad [0, 0], [-2, -2]]]], [[1, 4, 2], [regular\_chain, [[-1, -1], [0, 0], [-1, -1]]]], [[1, 4, 3],$

$\quad [regular\_chain, [[-1, -1], [0, 0], [0, 0]]]], [[1, 4, 4], [regular\_chain, [[-1, -1], [0,$

$\quad 0], [1, 1]]]], [[1, 4, 5], [regular\_chain, [[-1, -1], [0, 0], [2, 2]]]], [[1, 5, 1],$

$\quad [regular\_chain, [[-1, -1], [1, 1], [-3, -3]]]], \Big[[1, 5, 2], \Big[regular\_chain, \Big[[-1, -1],$

$\quad [1, 1], \Big[-\dfrac{1482911}{1048576}, -\dfrac{741455}{524288}\Big]\Big]\Big]\Big], \Big[[1, 5, 3], \Big[regular\_chain, \Big[[-1, -1], [1, 1], \Big[$

$\quad -\dfrac{741455}{1048576}, -\dfrac{741455}{1048576}\Big]\Big]\Big]\Big], [[1, 5, 4], [regular\_chain, [[-1, -1], [1, 1], [0, 0]]]],$

$\quad \Big[[1, 5, 5], \Big[regular\_chain, \Big[[-1, -1], [1, 1], \Big[\dfrac{741455}{1048576}, \dfrac{741455}{1048576}\Big]\Big]\Big]\Big], \Big[[1, 5, 6],$

$\quad \Big[regular\_chain, \Big[[-1, -1], [1, 1], \Big[\dfrac{741455}{524288}, \dfrac{1482911}{1048576}\Big]\Big]\Big]\Big], [[1, 5, 7], [regular\_chain,$

$\quad [[-1, -1], [1, 1], [3, 3]]]], [[2, 1, 1], [regular\_chain, [[0, 0], [-1, -1], [-2, -2]]]],$

$[[2, 1, 2], [regular\_chain, [[0, 0], [-1, -1], [-1, -1]]]], [[2, 1, 3], [regular\_chain,$

$[[0, 0], [-1, -1], [0, 0]]]], [[2, 1, 4], [regular\_chain, [[0, 0], [-1, -1], [1, 1]]]], [[2,$

$1, 5], [regular\_chain, [[0, 0], [-1, -1], [2, 2]]]], [[2, 2, 1], [regular\_chain, [[0, 0], [0,$

$0], [-1, -1]]]], [[2, 2, 2], [regular\_chain, [[0, 0], [0, 0], [0, 0]]]], [[2, 2, 3],$

$[regular\_chain, [[0, 0], [0, 0], [1, 1]]]], [[2, 3, 1], [regular\_chain, [[0, 0], [1, 1], [-2,$

$-2]]]], [[2, 3, 2], [regular\_chain, [[0, 0], [1, 1], [-1, -1]]]], [[2, 3, 3],$

$[regular\_chain, [[0, 0], [1, 1], [0, 0]]]], [[2, 3, 4], [regular\_chain, [[0, 0], [1, 1], [1,$

$1]]]], [[2, 3, 5], [regular\_chain, [[0, 0], [1, 1], [2, 2]]]], [[3, 1, 1], [regular\_chain,$

$[[1, 1], [-1, -1], [-3, -3]]]], \left[[3, 1, 2], \left[regular\_chain, \left[[1, 1], [-1, -1], \left[\right.\right.\right.$

$\left.\left.\left.-\frac{1482911}{1048576}, -\frac{741455}{524288}\right]\right]\right]\right], \left[[3, 1, 3], \left[regular\_chain, \left[[1, 1], [-1, -1], \left[-\frac{741455}{1048576},\right.\right.\right.\right.$

$\left.\left.\left.-\frac{741455}{1048576}\right]\right]\right]\right], [[3, 1, 4], [regular\_chain, [[1, 1], [-1, -1], [0, 0]]]], \left[[3, 1, 5],\right.$

$\left[regular\_chain, \left[[1, 1], [-1, -1], \left[\frac{741455}{1048576}, \frac{741455}{1048576}\right]\right]\right], \left[[3, 1, 6], \left[regular\_chain,\right.\right.$

$\left.\left[[1, 1], [-1, -1], \left[\frac{741455}{524288}, \frac{1482911}{1048576}\right]\right]\right], [[3, 1, 7], [regular\_chain, [[1, 1], [-1,$

$-1], [3, 3]]]], [[3, 2, 1], [regular\_chain, [[1, 1], [0, 0], [-2, -2]]]], [[3, 2, 2],$

$[regular\_chain, [[1, 1], [0, 0], [-1, -1]]]], [[3, 2, 3], [regular\_chain, [[1, 1], [0, 0],$

$[0, 0]]]], [[3, 2, 4], [regular\_chain, [[1, 1], [0, 0], [1, 1]]]], [[3, 2, 5], [regular\_chain,$

$[[1, 1], [0, 0], [2, 2]]]], \left[[3, 3, 1], \left[regular\_chain, \left[[1, 1], \left[\frac{1}{2}, \frac{1}{2}\right], [-2, -2]\right]\right]\right], \left[[3,\right.$

$3, 2], \left[regular\_chain, \left[[1, 1], \left[\frac{1}{2}, \frac{1}{2}\right], \left[-\frac{859611}{1048576}, -\frac{429805}{524288}\right]\right]\right], [[3, 3, 3],$

$\left[regular\_chain, \left[[1, 1], \left[\frac{1}{2}, \frac{1}{2}\right], [1, 1]\right]\right], [[3, 4, 1], [regular\_chain, [[1, 1], [1, 1], [$

$-1, -1]]]], [[3, 4, 2], [regular\_chain, [[1, 1], [1, 1], [0, 0]]]], \left[[3, 4, 3],\right.$

$\left[regular\_chain, \left[[1, 1], [1, 1], \left[\frac{1}{2}, \frac{1}{2}\right]\right]\right], [[3, 4, 4], [regular\_chain, [[1, 1], [1, 1],$

$[1, 1]]]], [[3, 4, 5], [regular\_chain, [[1, 1], [1, 1], [2, 2]]]], [[3, 5, 1], [regular\_chain,$

$$[[1, 1], [2, 2], [-3, -3]]]], \left[[3, 5, 2], \left[regular\_chain, \left[[1, 1], [2, 2], \left[-\frac{859611}{524288}, \right.\right.\right.$$

$$\left.\left.\left. -\frac{1719221}{1048576}\right]\right]\right]\right], [[3, 5, 3], [regular\_chain, [[1, 1], [2, 2], [0, 0]]]]\right]$$

<div align="center">59</div> **(16)**

> $Schur6 := eval(Schur, k = 4); Z6list := CADFull([expand(Schur6)], R[variables], output = list); nops(\%);$

$$Schur6 := a^4 (a - b) (a - c) + b^4 (b - a) (b - c) + c^4 (c - a) (c - b)$$

$$Z6list := \left[[[1, 1, 1], [regular\_chain, [[-1, -1], [-2, -2], [0, 0]]]], [[1, 2, 1], \right.$$

$$[regular\_chain, [[-1, -1], [-1, -1], [-2, -2]]]], [[1, 2, 2], [regular\_chain, [[-1, -1],$$

$$[-1, -1], [-1, -1]]]], \left[[1, 2, 3], \left[regular\_chain, \left[[-1, -1], [-1, -1], \left[-\frac{1}{2}, -\frac{1}{2}\right]\right]\right]\right],$$

$$[[1, 2, 4], [regular\_chain, [[-1, -1], [-1, -1], [0, 0]]]], [[1, 2, 5], [regular\_chain, [[$$

$$-1, -1], [-1, -1], [1, 1]]]], \left[[1, 3, 1], \left[regular\_chain, \left[[-1, -1], \left[-\frac{1}{2}, -\frac{1}{2}\right], [0,\right.\right.\right.$$

$$\left.\left.\left. 0]\right]\right]\right], [[1, 4, 1], [regular\_chain, [[-1, -1], [0, 0], [-2, -2]]]], [[1, 4, 2],$$

$$[regular\_chain, [[-1, -1], [0, 0], [-1, -1]]]], [[1, 4, 3], [regular\_chain, [[-1, -1],$$

$$[0, 0], [0, 0]]]], [[1, 5, 1], [regular\_chain, [[-1, -1], [1, 1], [0, 0]]]], [[2, 1, 1],$$

$$[regular\_chain, [[0, 0], [-1, -1], [-2, -2]]]], [[2, 1, 2], [regular\_chain, [[0, 0], [-1,$$

$$-1], [-1, -1]]]], [[2, 1, 3], [regular\_chain, [[0, 0], [-1, -1], [0, 0]]]], [[2, 2, 1],$$

$$[regular\_chain, [[0, 0], [0, 0], [-1, -1]]]], [[2, 2, 2], [regular\_chain, [[0, 0], [0, 0],$$

$$[0, 0]]]], [[2, 2, 3], [regular\_chain, [[0, 0], [0, 0], [1, 1]]]], [[2, 3, 1], [regular\_chain,$$

$$[[0, 0], [1, 1], [0, 0]]]], [[2, 3, 2], [regular\_chain, [[0, 0], [1, 1], [1, 1]]]], [[2, 3, 3],$$

$$[regular\_chain, [[0, 0], [1, 1], [2, 2]]]], [[3, 1, 1], [regular\_chain, [[1, 1], [-1, -1],$$

$$[0, 0]]]], [[3, 2, 1], [regular\_chain, [[1, 1], [0, 0], [0, 0]]]], [[3, 2, 2], [regular\_chain,$$

$$[[1, 1], [0, 0], [1, 1]]]], [[3, 2, 3], [regular\_chain, [[1, 1], [0, 0], [2, 2]]]], \left[[3, 3, 1],\right.$$

$$\left[regular\_chain, \left[[1, 1], \left[\frac{1}{2}, \frac{1}{2}\right], [0, 0]\right]\right]\right], [[3, 4, 1], [regular\_chain, [[1, 1], [1, 1], [$$

$-1, -1\,]\,]\,]\,]$, $[\,[\,3, 4, 2\,], [\,regular\_chain, [\,[\,1, 1\,], [\,1, 1\,], [\,0, 0\,]\,]\,]\,]$, $\Big[\,[\,3, 4, 3\,],$

$\Big[\,regular\_chain, \Big[\,[\,1, 1\,], [\,1, 1\,], \Big[\,\dfrac{1}{2}, \dfrac{1}{2}\,\Big]\,\Big]\,\Big]\,\Big]$, $[\,[\,3, 4, 4\,], [\,regular\_chain, [\,[\,1, 1\,], [\,1, 1\,],$

$[\,1, 1\,]\,]\,]\,]$, $[\,[\,3, 4, 5\,], [\,regular\_chain, [\,[\,1, 1\,], [\,1, 1\,], [\,2, 2\,]\,]\,]\,]$, $[\,[\,3, 5, 1\,], [\,regular\_chain,$

$[\,[\,1, 1\,], [\,2, 2\,], [\,0, 0\,]\,]\,]\,]\,]$

$$31 \tag{17}$$

> $Schur1 := eval(Schur, k = 1); Z1list := CADFull([\,expand(Schur1)\,], R[\,variables\,], output$
    $= list); nops(\%);$
    $$Schur1 := a\,(a - b)\,(a - c) + b\,(b - a)\,(b - c) + c\,(c - a)\,(c - b)$$

Warning, no method was specified, McCallum's algorithm will be
used

$Z1list := \Big[\,[\,[\,1, 1, 1\,], [\,regular\_chain, [\,[\,-1, -1\,], [\,-2, -2\,], [\,-1, -1\,]\,]\,]\,]$, $\Big[\,[\,1, 1, 2\,],$

$\Big[\,regular\_chain, \Big[\,[\,-1, -1\,], [\,-2, -2\,], \Big[\,\dfrac{806661}{1048576}, \dfrac{403331}{524288}\,\Big]\,\Big]\,\Big]\,\Big]$, $[\,[\,1, 1, 3\,],$

$[\,regular\_chain, [\,[\,-1, -1\,], [\,-2, -2\,], [\,2, 2\,]\,]\,]\,]$, $[\,[\,1, 2, 1\,], [\,regular\_chain, [\,[\,-1, -1\,], [$

$-1, -1\,], [\,-2, -2\,]\,]\,]\,]$, $[\,[\,1, 2, 2\,], [\,regular\_chain, [\,[\,-1, -1\,], [\,-1, -1\,], [\,-1, -1\,]\,]\,]\,]$, $\Big[\,[\,1,$

$2, 3\,], \Big[\,regular\_chain, \Big[\,[\,-1, -1\,], [\,-1, -1\,], \Big[\,-\dfrac{1}{2}, -\dfrac{1}{2}\,\Big]\,\Big]\,\Big]\,\Big]$, $[\,[\,1, 2, 4\,], [\,regular\_chain, [\,[$

$-1, -1\,], [\,-1, -1\,], [\,0, 0\,]\,]\,]\,]$, $[\,[\,1, 2, 5\,], [\,regular\_chain, [\,[\,-1, -1\,], [\,-1, -1\,], [\,1, 1\,]\,]\,]\,]$,

$\Big[\,[\,1, 3, 1\,], \Big[\,regular\_chain, \Big[\,[\,-1, -1\,], \Big[\,-\dfrac{1}{2}, -\dfrac{1}{2}\,\Big], [\,-1, -1\,]\,\Big]\,\Big]\,\Big]$, $\Big[\,[\,1, 3, 2\,],$

$\Big[\,regular\_chain, \Big[\,[\,-1, -1\,], \Big[\,-\dfrac{1}{2}, -\dfrac{1}{2}\,\Big], \Big[\,\dfrac{201665}{524288}, \dfrac{403331}{1048576}\,\Big]\,\Big]\,\Big]\,\Big]$, $\Big[\,[\,1, 3, 3\,],$

$\Big[\,regular\_chain, \Big[\,[\,-1, -1\,], \Big[\,-\dfrac{1}{2}, -\dfrac{1}{2}\,\Big], [\,2, 2\,]\,\Big]\,\Big]\,\Big]$, $[\,[\,1, 4, 1\,], [\,regular\_chain, [\,[\,-1, -1\,],$

$[\,0, 0\,], [\,-2, -2\,]\,]\,]\,]$, $[\,[\,1, 4, 2\,], [\,regular\_chain, [\,[\,-1, -1\,], [\,0, 0\,], [\,-1, -1\,]\,]\,]\,]$, $[\,[\,1, 4, 3\,],$

$[\,regular\_chain, [\,[\,-1, -1\,], [\,0, 0\,], [\,0, 0\,]\,]\,]\,]$, $[\,[\,1, 4, 4\,], [\,regular\_chain, [\,[\,-1, -1\,], [\,0,$

$0\,], [\,1, 1\,]\,]\,]\,]$, $[\,[\,1, 4, 5\,], [\,regular\_chain, [\,[\,-1, -1\,], [\,0, 0\,], [\,2, 2\,]\,]\,]\,]$, $[\,[\,1, 5, 1\,],$

$[\,regular\_chain, [\,[\,-1, -1\,], [\,1, 1\,], [\,-4, -4\,]\,]\,]\,]$, $\Big[\,[\,1, 5, 2\,], \Big[\,regular\_chain, \Big[\,[\,-1, -1\,],$

$[\,1, 1\,], \Big[\,-\dfrac{146543}{65536}, -\dfrac{2344687}{1048576}\,\Big]\,\Big]\,\Big]\,\Big]$, $\Big[\,[\,1, 5, 3\,], \Big[\,regular\_chain, \Big[\,[\,-1, -1\,], [\,1, 1\,], \Big[$

$-\dfrac{2344687}{2097152}, -\dfrac{2344687}{2097152}\,\Big]\,\Big]\,\Big]\,\Big]$, $[\,[\,1, 5, 4\,], [\,regular\_chain, [\,[\,-1, -1\,], [\,1, 1\,], [\,0, 0\,]\,]\,]\,]$,

$\Big[\,[\,1, 5, 5\,], \Big[\,regular\_chain, \Big[\,[\,-1, -1\,], [\,1, 1\,], \Big[\,\dfrac{2344687}{2097152}, \dfrac{2344687}{2097152}\,\Big]\,\Big]\,\Big]\,\Big]$, $\Big[\,[\,1, 5, 6\,],$

$$\left[regular\_chain, \left[[-1, -1], [1, 1], \left[\frac{2344687}{1048576}, \frac{146543}{65536}\right]\right]\right]\right], [[1, 5, 7], [regular\_chain,$$

$$[[-1, -1], [1, 1], [4, 4]]]], [[2, 1, 1], [regular\_chain, [[0, 0], [-1, -1], [-2, -2]]]],$$

$$[[2, 1, 2], [regular\_chain, [[0, 0], [-1, -1], [-1, -1]]]], [[2, 1, 3], [regular\_chain,$$

$$[[0, 0], [-1, -1], [0, 0]]]], [[2, 1, 4], [regular\_chain, [[0, 0], [-1, -1], [1, 1]]]], [[2,$$

$$1, 5], [regular\_chain, [[0, 0], [-1, -1], [2, 2]]]], [[2, 2, 1], [regular\_chain, [[0, 0], [0,$$

$$0], [-1, -1]]]], [[2, 2, 2], [regular\_chain, [[0, 0], [0, 0], [0, 0]]]], [[2, 2, 3],$$

$$[regular\_chain, [[0, 0], [0, 0], [1, 1]]]], [[2, 3, 1], [regular\_chain, [[0, 0], [1, 1], [-2,$$

$$-2]]]], [[2, 3, 2], [regular\_chain, [[0, 0], [1, 1], [-1, -1]]]], [[2, 3, 3],$$

$$[regular\_chain, [[0, 0], [1, 1], [0, 0]]]], [[2, 3, 4], [regular\_chain, [[0, 0], [1, 1], [1,$$

$$1]]]], [[2, 3, 5], [regular\_chain, [[0, 0], [1, 1], [2, 2]]]], [[3, 1, 1], [regular\_chain,$$

$$[[1, 1], [-1, -1], [-4, -4]]]], \left[[3, 1, 2], \left[regular\_chain, \left[[1, 1], [-1, -1], \right[\right.\right.$$

$$\left.\left.-\frac{146543}{65536}, -\frac{2344687}{1048576}\right]\right]\right]\right], \left[[3, 1, 3], \left[regular\_chain, \left[[1, 1], [-1, -1], \left[-\frac{2344687}{2097152}, \right.\right.\right.$$

$$\left.\left.\left.-\frac{2344687}{2097152}\right]\right]\right]\right], [[3, 1, 4], [regular\_chain, [[1, 1], [-1, -1], [0, 0]]]], \left[[3, 1, 5], \right.$$

$$\left[regular\_chain, \left[[1, 1], [-1, -1], \left[\frac{2344687}{2097152}, \frac{2344687}{2097152}\right]\right]\right]\right], \left[[3, 1, 6], \left[regular\_chain, \right.\right.$$

$$\left.\left[[1, 1], [-1, -1], \left[\frac{2344687}{1048576}, \frac{146543}{65536}\right]\right]\right]\right], [[3, 1, 7], [regular\_chain, [[1, 1], [-1,$$

$$-1], [4, 4]]]], [[3, 2, 1], [regular\_chain, [[1, 1], [0, 0], [-2, -2]]]], [[3, 2, 2],$$

$$[regular\_chain, [[1, 1], [0, 0], [-1, -1]]]], [[3, 2, 3], [regular\_chain, [[1, 1], [0, 0],$$

$$[0, 0]]]], [[3, 2, 4], [regular\_chain, [[1, 1], [0, 0], [1, 1]]]], [[3, 2, 5], [regular\_chain,$$

$$[[1, 1], [0, 0], [2, 2]]]], \left[[3, 3, 1], \left[regular\_chain, \left[[1, 1], \left[\frac{1}{2}, \frac{1}{2}\right], [-2, -2]\right]\right]\right], \left[[3, \right.$$

$$3, 2], \left[regular\_chain, \left[[1, 1], \left[\frac{1}{2}, \frac{1}{2}\right], \left[-\frac{403331}{1048576}, -\frac{201665}{524288}\right]\right]\right]\right], \left[[3, 3, 3], \right.$$

$$\left[regular\_chain, \left[[1, 1], \left[\frac{1}{2}, \frac{1}{2}\right], [1, 1]\right]\right]\right], [[3, 4, 1], [regular\_chain, [[1, 1], [1, 1], [$$

$$-1, -1]]]], [[3, 4, 2], [regular\_chain, [[1, 1], [1, 1], [0, 0]]]], \left[[3, 4, 3], \right.$$

$$\left[regular\_chain, \left[[1, 1], [1, 1], \left[\frac{1}{2}, \frac{1}{2} \right] \right] \right] \right], [[3, 4, 4], [regular\_chain, [[1, 1], [1, 1],$$

$$[1, 1]]]], [[3, 4, 5], [regular\_chain, [[1, 1], [1, 1], [2, 2]]]], [[3, 5, 1], [regular\_chain,$$

$$[[1, 1], [2, 2], [-2, -2]]]], \left[[3, 5, 2], \left[regular\_chain, \left[[1, 1], [2, 2], \left[-\frac{403331}{524288}, \right. \right. \right.$$

$$\left. \left. \left. -\frac{806661}{1048576} \right] \right] \right] \right], [[3, 5, 3], [regular\_chain, [[1, 1], [2, 2], [1, 1]]]] \right]$$

59 **(18)**

>